Chaotic Nature of Logistic Map

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Introduction

There is an important subclass of systems, the so-called discrete dynamical systems, in which the time variable can be treated as a discrete variable rather than a continuous one. This may mean, for example, that it is sufficient or meaningful to measure certain physical variables after finite intervals of time, say an hour, a week, a month, etc. rather than on a continuous basis. Examples of this type of systems are: population of an insect species in a forest, radioactive decay, rain fall, temperature in a city. These systems are often represented by difference equations/recursion relations/iterated maps or simply maps. These maps could be of any dimensions depending on the number of physical variables. Now we will study one discrete dynamical system called Logistic map.

Some basic concepts

First we will see few important concepts of a dynamical system.

Orbit

Given a map f and a point x in the domain of f, we call the set of points $\{x, f(x), f^2(x), ...\}$ the orbit of x under f.

Fixed point

Given a point p in the domain of f, if $f(p) = p$ then we call p a **fixed point** of the map f.

Period and Periodic point

We say that a point x_0 is periodic of period n if $f^n(x_0) = x_0$ for some $n > 0$.
We call the orbit of x_0 a **periodic orbit** or **cycle** in this case.

The Logistic map has a quadratic non linearity and is represented by the equation

$$x_{n+1} = ax_n(1 - x_n)(= f(x_n)), \qquad n = 0, 1, 2, ...$$

where a is a parameter and we assume that $0 \leq x \leq 1$. As shown in Fig 2.1 the graph of $f(x_n)$ is a parabola with a maximum value $\frac{a}{4}$ at $x = \frac{1}{2}$. Now depending on the value of the parameter a we will observe the nature of this map.

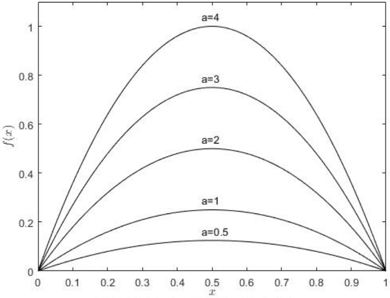


Fig. 2.1. Graph of $f(x)$ for the logistic map

The Logistic Map

Now we will investigate the behaviours of the Logistic map as the parameter a varied.

Proposition

For $a \leq 1$, 0 is the only fixed point and it is stable for $a < 1$ and unstable for $a > 1$. For $a > 1$, there exists an additional fixed point $1 - \frac{1}{a}$ which is stable for $1 < a < 3$ and unstable for $a > 3$.

Observations

For $0 \leq a < 4$ we observe the following results:

1. When $a = 1$ transcritical bifurcation occurs.
2. If $1 < a < 3$, then 0 is repelling fixed point and $1 - \frac{1}{a}$ is attracting fixed point.
3. When $a = 3$ period doubling bifurcation occurs.
4. When $a \approx 3.45$ another period doubling bifurcation occurs.
5. In the interval $3.44 < a < 4$ changes occur rapidly.

The Logistic Map

In the interval $0 \le a < 4$, the bifurcation diagram of the logistic map is the following:

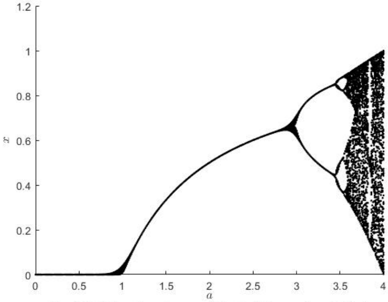


Fig. 2.2. Bifurcation diagram of the logistic map for $a \in (0, 4)$

Now we will see what happens when $a = 4$. The next goal is to prove that the logistic map is **chaotic** for $a = 4$. To prove this we will use the concept of 'Topological Conjugacy'.

Topological Conjugacy

Let $f : D \longrightarrow D$ and $g : E \longrightarrow E$ be functions. Then f is topologically conjugate to g if there is a homeomorphism $\tau : D \longrightarrow E$ such that $\tau \circ f = g \circ \tau$. In this case, τ is called a topological conjugacy.
We represent this relationship by the following commutative diagrams:

$$\begin{array}{ccc} D & \xrightarrow{f} & D \\ \tau \downarrow & & \downarrow \tau \\ E & \xrightarrow[g]{} & E \end{array} \qquad \text{or} \qquad \begin{array}{ccc} x & \xrightarrow{f} & f(x) \\ \tau \downarrow & & \downarrow \tau \\ \tau(x) & \xrightarrow[g]{} & g(\tau(x)) = \tau(f(x)) \end{array}$$

Now, at first we will discuss the definition of chaotic maps and we will show that if two maps f and g are topologically conjugate, then f is chaotic if and only if g is chaotic.

Topologically Transitive

The function $f : D \longrightarrow D$ is topologically transitive if for all open sets U and V in D, there is x in U and a natural number n such that $f^n(x)$ is in V, i.e., $f^n(U) \cap V \neq \phi$.

This is to say that if we take two open sets U and V in the domain and look far enough in the orbits of all of the elements of U under f we will eventually find some element in V. For this reason sometimes we describe the action of the function which is topologically transitive as **mixing the domain**.

Sensitive Dependence on Initial Condition

Let D be a metric space with metric d. The function $f : D \longrightarrow D$ has sensitive dependence on initial conditions if $\exists\delta > 0$ such that, for any $x \in D$ and any open set N containing x, $\exists y \in N$ and an $n \geq 0$ such that $d(f^n(x), f^n(y)) > \delta$.

Practically speaking, f has sensitive dependence means even **the slightest change in initial conditions may result in dramatically different results in values under iteration of the function.**

Our third property requires no new definition. It is simply that *periodic points are dense under* f. Essentially, this means that **wherever we look in the domain, we will find a periodic point under** f.

With all of the above three properties we can finally formalize our definition of **Chaos**:

Chaotic function

Let (X, d) be a metric space and let $V \subseteq X$. We say that $f : V \longrightarrow V$ is chaotic on V if

1. f is topologically transitive,
2. f has sensitive dependence on initial conditions,
3. Periodic points under f are dense in V.

Properties Shared by Topologically Conjugate Maps

We will start by exploring the relationships between the dynamics of $x \in D$ under f and the dynamics of $\tau(x) \in E$ under g, where the maps f and g are topologically conjugate.

Theorem 1

Suppose that $f : D \longrightarrow D$ and $g : E \longrightarrow E$ are topologically conjugate under the homeomorphism $\tau : D \longrightarrow E$. Then a point p in D is a fixed point under f iff $\tau(p)$ is a fixed point under g.

Theorem 2

Let, $f : D \longrightarrow D$ and $g : E \longrightarrow E$ are topologically conjugate under the homeomorphism $\tau : D \longrightarrow E$. Then a point $x \in D$ is a periodic point of period n under f iff $\tau(x)$ is a periodic point of period n under g.

Properties Shared by Topologically Conjugate Maps

Theorem

Let (X, d_1) and (Y, d_2) be metric spaces and let $f : X \longrightarrow X$ and $g : Y \longrightarrow Y$ be two maps with a topological conjugacy $\tau : X \longrightarrow Y$ between them. Then f is topologically transitive if and only if g is topologically transitive.

Proof.

Suppose that f is topologically transitive and let U and V be two open sets in Y . We know that $\tau^{-1}(U)$ and $\tau^{-1}(V)$ are both open in X. Since f is topologically transitive, we know that $\exists k \in \mathbb{N}$ such that

$$(f^k \circ \tau^{-1}(U)) \cap \tau^{-1}(V) \neq \phi.$$

We see that, $(\tau \circ f^k \circ \tau^{-1})(U) = (g^k \circ \tau \circ \tau^{-1})(U) = g^k(U)$. So,

$$g^k(U) \cap V \neq \phi.$$

Hence g is topologically transitive. Similarly the converse part holds. □

Properties Shared by Topologically Conjugate Maps

Theorem

Let (X, d_1) and (Y, d_2) be metric spaces and let $f : X \longrightarrow X$ and $g : Y \longrightarrow Y$ be two maps with a topological conjugacy $\tau : X \longrightarrow Y$ between them. Then periodic points are dense in X under f iff periodic points are dense in Y under g.

Proof.

Suppose that periodic points are dense in X under f. Let U be an open set in Y. We then know that $\tau^{-1}(U)$ is an open set in X. Since periodic points are dense in X under f so there exists a periodic point $p \in \tau^{-1}(U)$ under f. Since p is a periodic point under f, $\tau(p)$ is a periodic point under g. Since $\tau(p) \in U$, so, every open set in Y contains a periodic point under g. Suppose that periodic points are dense in Y under g. Let U be an open set in X. So, $\tau(U)$ is an open set in Y . Since periodic points are dense in Y under g so there exists a periodic point $p \in \tau(U)$ under g. Since p is a periodic point under g, $\tau^{-1}(p)$ is a periodic point under f. Since $\tau^{-1}(p) \in U$, so, every open set in X contains a periodic point under f. □

Properties Shared by Topologically Conjugate Maps

Theorem

Let (X, d_1) and (Y, d_2) be metric spaces and let $f : X \longrightarrow X$ and $g : Y \longrightarrow Y$ be two maps with a topological conjugacy $\tau : X \longrightarrow Y$ between them. Then f has sensitive dependence on initial conditions if and only if g has sensitive dependence on initial conditions.

Proof.

Suppose that f has sensitive dependence on initial conditions and let $\epsilon > 0$ be the real number guaranteed to exist by that property. Let $x \in Y$ and let N be a neighbourhood of x. We know that there exists some point $y \in \tau^{-1}(N) \subseteq X$ and some positive integer n such that

$$d_1((f^n \circ \tau^{-1}(x), f^n(y)) > \epsilon$$

Since τ^{-1} is continuous we know that there exists some $\delta > 0$ such that $d_1(\tau^{-1}(s), \tau^{-1}(t)) < \epsilon$ if $d_2(s, t) \leq \delta$. Note that since ϵ is fixed, δ is as well.

Proof ctd.

Now if $d_2(g^n(x), (g^n \circ \tau)(y)) \leq \delta$. Then $d_1((\tau^{-1} \circ g^n)(x), (\tau^{-1} \circ g^n \circ \tau)(y)) = d_1((f^n \circ \tau^{-1})(x), f^n(y)) < \epsilon$, this is a contradiction. So, $d_2(g^n(x), (g^n \circ \tau)(y)) > \delta$. Thus we have shown that given any point $x \in Y$ and any neighbourhood N around it, we can find a point $w \in N$ and and a positive integer n such that $d_2(g^n(x), g^n(w)) > \delta$. As such, g has sensitive dependence on initial conditions.

Similarly using the continuity of τ we can prove the converse part. □

Theorem

Let two maps f and g are topologically conjugate. Then f is chaotic if and only if g is chaotic.

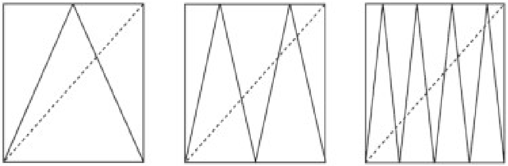
Proof.

Using the above three theorems and the definition of chaotic map, it is quite obvious. □

Sometimes it is hard to prove that a map f is chaotic using direct definition. In such cases we want to find a map g which is topologically conjugate to f and then if we can prove that g is chaotic then it is also true that f is chaotic. Now, we will be doing exactly same with the Logistic map $f : [0,1] \longrightarrow [0,1]$ defined by $f(x) = ax(1-x)$, where $a = 4$, i.e., $f(x) = 4x(1-x)$ and the Tent map $g : [0,1] \longrightarrow [0,1]$ defined by

$$g(x) = \left\{ \begin{array}{ll} 2x, & \text{if } 0 \leq x < \frac{1}{2} \\ -2x+2, & \text{if } \frac{1}{2} \leq x \leq 1 \end{array} \right\}$$

The graph of first, second and third iterate of the Tent map is the following:



Tent map is chaotic

Theorem

Tent map is chaotic in $[0,1]$.

Proof.

1) [Density of Periodic points] g^n maps each interval $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$ to $[0,1]$ for $k = 0, 1, \ldots, 2^n - 1$. So, g^n intersects the line $y = x$ once in each interval. As a result, each interval contains a fixed point of g^n or equivalently a periodic point of g of period n. So, periodic points of g are dense in $[0,1]$.

2) [Transitivity] Let, U_1 and U_2 be open subintervals of $[0,1]$. For n sufficiently large and for some k, U_1 contains an interval of the form $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$. Therefore, g^n maps U_1 to $[0,1]$ which contains U_2.

3) [Sensitive dependence on initial conditions] Let $x_0 \in [0,1]$. We will show that a sensitivity constant $\beta = \frac{1}{2}$ works. As in 2), any open interval U about x_0 is mapped by g^n to $[0,1]$ for some sufficiently large n. Therefore, there exists $y_0 \in U$ such that $|g^n(x_0) - g^n(y_0)| \geq \frac{1}{2} = \beta$. □

Relation Between Tent map and Logistic map

Now if we can show that the Tent map g is topological conjugate of Logistic map f, then we are done.

The function $\tau : [0,1] \longrightarrow [0,1]$ defined by $\tau(x) = \frac{1}{2}(1 - \cos(\pi x))$ is a homeomorphism.

Given function is,

$$\tau(x) = \frac{1}{2}(1 - \cos(\pi x)), x \in [0,1].$$

Since, cos function is continuous, so, τ is continuous.
Let $x, y \in [0,1]$ and $\tau(x) = \tau(y) \Rightarrow \cos(\pi x) = \cos(\pi y) \Rightarrow x = y$.
Therefore, τ is injective.
Now, let $y \in [0,1]$ and $\tau(x) = y$.
$\therefore \frac{1}{2}(1 - \cos(\pi x)) = y \Rightarrow x = \frac{1}{\pi}\cos^{-1}(1 - 2y)$. Therefore, $\forall y \in [0,1]$ we get an $x \in [0,1]$. Hence τ is surjective.
$\therefore \tau$ is a bijection and $\tau^{-1} : [0,1] \longrightarrow [0,1]$ defined by $\tau^{-1}(x) = \frac{1}{\pi}\cos^{-1}(1 - 2y)$. So, τ^{-1} is also continuous.
Hence τ is a homeomorphism.

Relation Between Tent map and Logistic map

Theorem

Tent map & Logistic map ($a = 4$) are topologically conjugate.

Proof.

Tent map is $g : [0,1] \longrightarrow [0,1]$ defined by

$$g(x) = \begin{cases} 2x, & \text{if } 0 \leq x < \frac{1}{2} \\ -2x + 2, & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

Logistic map is $f : [0,1] \longrightarrow [0,1]$ defined by

$$f(x) = 4x(1-x)$$

Consider the function $\tau : [0,1] \longrightarrow [0,1]$ defined by

$$\tau(x) = \frac{1}{2}(1 - \cos(\pi x)) = \sin^2(\frac{\pi x}{2}).$$

When $x \in [0, \frac{1}{2})$, we get, $(\tau \circ g)(x) = \tau(2x) = \sin^2(\pi x)$

Proof ctd.

and

$$\begin{aligned}(f \circ \tau)(x) = f\left(\sin^2\left(\frac{\pi x}{2}\right)\right) &= 4\sin^2\left(\frac{\pi x}{2}\right)\left(1-\sin^2\left(\frac{\pi x}{2}\right)\right) \\ &= 4\sin^2\left(\frac{\pi x}{2}\right)\cos^2\left(\frac{\pi x}{2}\right) \\ &= \left(2\sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi x}{2}\right)\right)^2 \\ &= \sin^2(\pi x)\end{aligned}$$

So, we get $(\tau \circ g) = (f \circ \tau), \forall x \in [0, \frac{1}{2})$.
Now when $x \in [\frac{1}{2}, 1]$, we get,

$$(\tau \circ g)(x) = \tau(2(1-x)) = \sin^2(\pi(1-x)) = \sin^2(\pi x) = (f \circ \tau)(x).$$

□

Proof ctd.

$$\therefore (\tau \circ g) = (f \circ \tau), \forall x \in [0,1].$$

In the above, we also proved that τ is also a homeomorphism, so, τ is a topological conjugacy between the Tent map and Logistic map. Hence, Tent map and Logistic map are topologically conjugate. □

Theorem

The Logistic map, $f : [0,1] \longrightarrow [0,1]$ is defined by $f(x) = ax(1-x)$ is chaotic for $a = 4$.

Proof.

We know that if two maps f and g are topologically conjugate, then f is chaotic iff g is chaotic.
Now we have proved that, for $a = 4$ Logistic map and Tent map are topologically conjugate and Tent map is chaotic, so, Logistic map is also chaotic for $a = 4$. □

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Thank You!